Eells-Sampson-Hartman's Existence Theorem

The singular goal of these notes is to give a proof of the following theorem:

Theorem: (Eells-Sampson)

Let M, N be smooth, closed Riemannian manifolds, and assume $\sec_N \leq 0$.

Let $f_0 \in C^\infty(M,N)$ be an arbitrary smooth map, and let $[f_0]$ denote its homotopy class in $C^\infty(M,N)$.

Then there is a harmonic map $f_\infty \in [f_0]$.

A motivating example: Recall the IVP problem for the scalar heat eqtn on the circle:

$$\begin{cases} \partial_t u = u_{xx} & \text{on } S^1 \times [0,T) \\ u(x,0) = f(x) & \text{on } S^1 \end{cases}$$

Here, the initial data can be taken in $C^0(S^1)$, for instance. One can show that the HE deforms the initial data f through smooth (analytic) functions for all time, and that as $t \to \infty$, $u(x,t) \to \bar{u} = \int_{S^1} f(x)dx$.

(**physical intuition:** as $t \to \infty$, an initial heat distribution homogenizes, and the heat everywhere converges to the average initial heat — assuming conservation of energy in the circle S^1). N.B. the constant functions are exactly the harmonic functions on S^1, so the HE deforms an initial function to a a harmonic function.

Variationally: Harmonic functions are the stationary points of the Dir. energy functional. This energy has an associated gradient flow, i.e. a flow along which one deforms a function to decrease it's energy as efficiently as possible. For harmonic functions, this flow is given by the HE as above. In general, following a gradient flow gives us a way to hunt

for stationary points of functionals – Eells-Sampson carried this out to locate harmonic maps between manifolds, as a far reaching generalization of the above toy example.

Here is a setup, and an outline of the proof:

Recall: If $f:(M^m,g)\to(N^n,h)$ is a C^∞ map,

- $df\in\Gamma(T^*M\otimes f^*TN)$ (i.e. locally $df=\frac{\partial f^\alpha}{\partial x^i}dx^i\otimes\partial_\alpha|_f$)
- This bdl has an inner product so that
$$|df|^2=\langle df,df\rangle=\partial_i f^\alpha\cdot\partial_j f^\beta\cdot g^{ij}h_{\alpha\beta}|_f$$
- The energy density $e=e(f):M\to\mathbb{R}$ is defined by $e(x)=\frac{1}{2}|df_x|^2$.
- The Dirichlet energy of f is defined by $E(f)=\int_M e(f)=\frac{1}{2}\int_M|df|^2$

Main Def'n: Maps which are stationary points for $E(-)$ are called harmonic maps (or more generally, maps which solve the associated Euler-Lagrange equation $\Delta_{g,h}f=\tau(f)=\mathrm{tr}_g(\nabla df)=0$ for $E(-)$).

Goal: Find a way to smoothly homotope a given C^∞ map $f_0:M\to N$ to such a harmonic map $f_\infty:M\to N$.

Idea: Deform f_0 along the gradient flow of $E(-)$.

Step 1: Find the gradient flow of $E(-)$ (harmonic map heat flow)

Step 2: Show HMHF has short time existence (standard parabolic theory)

Step 3: Show HMHF exists for all time

- Show $e(f_t)$ is a subsoln of the HE (uses $\sec^N\le 0$)
- Do Moser iteration to get L^∞ control of $e(f_t)$
- Bootstrap to control all derivatives of f_t along the flow, and use compactness to show that HMHF must be immortal.

Step 4: Show HMHF converges to a harmonic map f_∞ as $t \to \infty$

Step 5: Show $f_\infty \in [f_0]$.

Step 1:

Recall: The 1st Variation Formula for $E(-)$ is given as follows: if $f: M \to N$, and $f_t: M \times (-\varepsilon, \varepsilon) \to N$ is a smooth variation with $f_0 = f$ and $\frac{\partial}{\partial t}|_{t=0} f_t = \varphi \in f^* TN$, then

$$\delta E(\varphi) = \frac{d}{dt}\Big|_{t=0} E(f_t) = -\int_M \langle \varphi, \tau(f) \rangle_h$$

Consequently, we see that the most efficient direction to vary f in so as to decrease $E(f)$ is the direction of its own tension field $\tau(f)$. Thus, if we can find a smooth family $f_t: M \to N$ solving the PDE

$$\frac{\partial}{\partial t} f_t = \tau(f_t) \quad \text{on} \quad M \times [0, T) \qquad (HMHF)$$

for some "max'l time of existence" $0 < T \leq \infty$, then for all $t \in [0, T)$ we have

$$\frac{d}{dt} E(f_t) = -\int_M |\tau(f_t)|^2 \leq 0$$

so that indeed $E(f_t)$ decreases monotonically along HMHF.

Step 2: (Very rough sketch): By using the Nash embedding theorem, we can isometrically embed (N, h) into $\mathbb{R}^N$. Written in this form, short time existence for the HMHF is translated into the corresponding problem for systems of parabolic PDEs, which is treated using classical methods.

The Punchline: $\exists$ some time of existence $T_0 \in (0, \infty]$ for the flow. We aim to show that $T_0 = \infty$ in the next step:

Step 3: The key to controlling the derivatives of f_t along HMHF is the following:

Lemma: When f_t evolves under the HMHF, the energy density $e = e(f_t)(x)$ evolves by

$$\partial_t e \leq \Delta e + C \cdot e$$

for some constant $C = C(M) > 0$. In particular, e is a subsolution of the heat equation on $M \times [0,T)$.

Proof: We aim to compute $(\partial_t - \Delta)e$ and see what happens....

Recall the Bochner Formula: if $f: M^m \to N^n$ is, and $\{e_i\}_{i=1}^m$ is an ON local frame of TM,

$$\Delta e = \tfrac{1}{2}\Delta |df|^2 = |\nabla^2 f|^2 + \underbrace{\sum_{i=1}^m \langle {}^f\nabla_{e_i}(\tau(f)), df(e_i)\rangle_N}_{(*)} + \sum_{i=1}^m \langle df(\mathrm{Ric}^M(e_i)), df(e_i)\rangle_N - \sum_{i=1,j=1}^{m\ n} R^N_m\left(df(e_i), df(e_j), df(e_j), df(e_i)\right)$$

Note that by the HMHF eqtn, $df(\partial_t) = \partial_t f = \tau(f)$, hence $(*)$ for us has the form

$$\sum_{i=1}^m \langle {}^f\nabla_{e_i}(\tau(f)), df(e_i)\rangle_N = \sum_{i=1}^m \langle {}^f\nabla_{e_i}(df(\partial_t)), df(e_i)\rangle$$

Next note that

$$\partial_t e = \tfrac{1}{2}\partial_t |df|^2 = \tfrac{1}{2}\sum_{i=1}^m \partial_t \langle df(e_i), df(e_i)\rangle = \sum_{i=1}^m \langle {}^f\nabla_{\partial_t}(df(e_i)), df(e_i)\rangle.$$

Using the fact that ${}^f\nabla$ is torsion free, and that $[\partial_t, e_i] = 0$, we have

$${}^f\nabla_{\partial_t}(df(e_i)) = {}^f\nabla_{e_i}(df(\partial_t))$$

and thus

$$(\partial_t - \Delta)e = -2|\nabla^2 f|^2 - \underbrace{\sum_{i=1}^m \langle df(\mathrm{Ric}^M(e_i)), df(e_i)\rangle_N}_{(1)} + \underbrace{\sum_{i=1,j=1}^{m\ n} R^N_m\left(df(e_i), df(e_j), df(e_j), df(e_i)\right)}_{(2)}.$$

If $\sec_N \leq 0$, then $(2) \leq 0$.

Since M is compact, $(1) \leq C(M)\cdot e$, since

$$\begin{aligned}\langle df(\mathrm{Ric}(e_i)), df(e_i)\rangle &= \langle df(g^{kl}\mathrm{Ric}_{li} e_k), df(e_i)\rangle \\ &= g^{kl}\mathrm{Ric}_{li}\langle df(e_k), df(e_i)\rangle \\ &\geq -C(\mathrm{Ric}^M)\cdot g_{li}\cdot g^{kl}\langle df(e_k), df(e_i)\rangle \\ &= -C(M)\cdot e\end{aligned}$$

Next, by using $(\partial_t - \Delta)e \leq 0$ together with the L^1 bound on e, we can bootstrap to obtain a $L^\infty(M)$ bound on e which decays in time

(uses ① $\partial_t f_t = \tau(f_t)$ and ② Sobolev ineq:— see Andrews-Hopper)

Then, using this boundedness, we obtain control on all the higher derivatives of f too:

> **Prop:** If $f: M \times [0,T) \longrightarrow N$ solves MCHF (with M, N cpct) with bdd $e(f)$, then $\exists\, C_k = C_k(\|e\|_{L^\infty(M\times[0,T))}, \|R^M\|_{C^k}, \|R^N\|_{C^k}) > 0$, $k = 1, 2, \dots$, s.t.
>
> $$\|\nabla^{(k)} df\| \leq C_k \cdot \left(1 + \frac{1}{t^{k/2}}\right)$$

Example: For $k=1$: We can compute the evolution of ∇df to obtain

$$\partial_t |\nabla df|^2 \leq \Delta |\nabla df|^2 - 2|\nabla^2 df|^2 + C_1 |\nabla df|^2 + C_2$$

where C_1, C_2 depend on $|\nabla^{(k)} R^M|, |\nabla^{(k)} R^N|, e$, for $k = 0, 1$.

We want to estimate $|\nabla df|^2$ from this, but we have no control on $|\nabla df|^2$ at $t=0$ to compare with in the max'm princ. So, instead, we set $F = t|\nabla df|^2 + |df|^2$, so that $F|_{t=0} = |df_0|^2 = 2e(f_0) \leq K$ for some $K < \infty$ by compactness. Moreover,

$$\partial_t F \leq \Delta F + (t \cdot C_1 - 1)|\nabla df|^2 + (t \cdot C_2 + C_3)$$

If $t \in (0, \frac{1}{C_1}]$, we thus can apply the max'm princ. to obtain

$$F = t|\nabla df|^2 + |df|^2 \leq \left(\tfrac{C_2}{2} t^2 + C_3 t + K^2\right) \leq C + K^2$$

and hence

$$t|\nabla df|^2 \leq C \quad \text{on } M \times [0, \tfrac{1}{C_1}].$$

We can then apply the same argument on $[\frac{1}{C_1}, \frac{2}{C_1}]$, ... etc, to get the same bound for **all** time.

The higher $k > 1$ case is by induction.

Finally, we show the flow is immortal: Let

$$T_\infty = \sup\{T : f(x,t) \in C^{2+\alpha,1+\alpha}(M\times[0,T], N)\}$$

and suppose for sake of contradiction that $T_\infty < \infty$. Then $f(x,t)$ exists and is unique (given $f(x,0)=f_0$) on $M\times[0,T_\infty)$.

Let $\beta<\alpha<1$. Note $f_t = f(x,t)$ is unif. bdd in $C^{2+\alpha}(M,N)$ with respect to t by assumption. Set

$$f_{T_\infty}(x) := f_0(x) + \int_0^{T_\infty} (\partial_t f)(x,t)\,dt$$

as a candidate extension of f_t from $M\times[0,T_\infty)$ to $M\times[0,T_\infty]$. Let $t_k \nearrow T_\infty$. Then $\{f_{t_k}\} \subseteq C^{2+\alpha}(M,N)$ is unif. bdd in k, and by Ascoli-Arzelà has a unique sublimit in $C^{2+\beta}(M,N)$, which must be $f_{T_\infty}: M\to N$. The above formula then shows that $f_{T_\infty} \in C^{2+\beta,0}(M,N)$.

Next we upgrade the regularity in t, using the HMHF $\partial_t f = \tau(f)$: Since $t\mapsto f_t \in C^{2,\beta}(M,N)$ is C^0 by the above, $t\mapsto \partial_t(f_t)$ is in fact a C^0 map into $C^{0,\beta}(M,N)$, as $\tau(f_t)\in C^{0,\beta}(M,N)$, enabling us to extend $t\mapsto f_t$ from $[0,T_\infty)$ to $[0,T_\infty]$.

Thus, $f\in C^{2+\beta,1+\beta}(M\times[0,T_\infty],N)$ (for arbitrary $\beta<\alpha<1$).

We can thus restart the HMHF from the data f_{T_∞} and use short time existence to push further, so that $f\in C^{2+\alpha,1+\alpha}(M\times[0,T_\infty+T_0],N)$, contradicting the defn of T_∞.

Thus $T_\infty = \infty$.

Step 4: Next, we show that $f_t \to f_\infty$ for a harmonic map f_∞.

Key Identity: From the energy evolution

$$\frac{d}{dt} E(f_t) = -\int_M |\tau(f_t)|^2 = -\|\tau(f_t)\|^2_{L^2(M)},$$

we compute that

$$-\int_0^\infty \|\tau(f_t)\|^2_{L^2(M)}\, dt = \lim_{t\nearrow\infty} E(f_t) - E(f_0)$$
$$\geq -E(f_0)$$

hence $\int_0^\infty \|\tau(f_t)\|^2_{L^2(M)}\, dt < E(f_0) < \infty$.

Thus, $\exists\, t_k \nearrow \infty$ s.t. $\|\tau(f_{t_k})\|_{L^2(M)} \to 0$ as $k \to \infty$.

Pairing this with the higher derivative bounds of Prop 3.6, it follows that we even have

$$\|\tau(f_{t_k})\|_{L^2(M)} \to 0 \quad \text{uniformly}.$$

By Ascoli-Arzela, we thus get that $f_{t_k} \xrightarrow{C^\infty} f_\infty$ with $\|\tau(f_\infty)\|_{L^2(M)} = 0$, where f_∞ is harmonic.

Step 5: For large enough k, we eventually have (by uniform convergence)

$$\sup_{x\in M} d_N(f_{t_k}(x), f_\infty(x)) < \text{inj}(N)$$

Thus, $\exists!$ min N-geodesic from $f_{t_k}(x)$ to $f_\infty(x)$, and it depends smoothly on $x \in M$ (since everything else is smooth). If we push f_{t_k} along these geodesics, we see that $[f_\infty] = [f_{t_k}]$.